

Superconductivity, Superfluidity and Holography

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Abstract. This is a concise review of holographic superconductors and superfluids. We highlight some predictions of the holographic models and the emphasis is given to physical aspects rather than to the technical details, although some references to understand the latter are systematically provided. We include gapped systems in the discussion, motivated by the physics of high-temperature superconductivity. In order to do so we consider a compactified extra dimension (with radius R), or, alternatively, a dilatonic field. The first setup can also be used to model cylindrical superconductors; when these are probed by an axial magnetic field a universal property of holography emerges: while for large R (compared to the other scales in the problem) non-local operators are suppressed, leading to the so called Little-Parks periodicity, the opposite limit shows non-local effects, e.g. the uplifting of the Little-Parks periodicity. This difference corresponds in the gravity side to a Hawking-Page phase transition.

1. Introduction: motivations, background material and conclusions

Holographic dualities have become useful tools to explore strongly coupled theories; in particular, in the last few years systems at finite temperature T and density of a $U(1)$ charge have been extensively studied. The main motivation comes from condensed matter physics (for a review see [1]): if the $U(1)$ charge is the electric charge and the $U(1)$ symmetry is spontaneously broken one identifies these systems with superconductors (SCs); when the $U(1)$ symmetry is global, instead, the spontaneously broken phase corresponds to superfluidity.

One of the main ideas behind the application of holography to superconductivity is the hope to shed light on the physics of unconventional SCs (such as high- T SCs), which are not completely described by the weakly coupled theory: the Bardeen-Cooper-Schrieffer theory. Cuprate high- T SCs, the prototype of unconventional materials, are obtained by doping a Mott insulator and as such they typically show in the (T versus “dopant concentration”) phase diagram an insulator phase close to the region where the $U(1)$ symmetry is spontaneously broken [2]. This calls for a way to describe both conductor/SC and insulator/SC transitions. Since any insulating material eventually conducts if it is probed by a strong enough external current, this in turn requires a method to study gapped phases.

In this article we will briefly review holographic SCs [3, 4] and superfluids (SFs) [5, 4]. The emphasis will be given to physical aspects rather than to the technical details, for which we will refer to the original papers. Given that all SF systems can be considered as SCs in the limit of non-dynamical electromagnetic (EM) gauge fields, for the sake of definiteness, we will use the terminology adopted in the literature of superconductivity, rather than that

of superfluidity (unless otherwise stated). We will focus on the best understood version of holographic duality, the anti de Sitter/conformal field theory (AdS/CFT) correspondence. The discussion of the previous paragraph then motivates us to introduce a conformal symmetry breaking in the infrared (IR), which is needed to have a gap. Moreover, since much of the interesting physics of cuprate SCs is layered, we shall restrict our analysis to 2+1 dimensional systems, dual to gravitational theories on four dimensional spacetimes. Although beyond the scope of this review, it is useful to mention that other regions of cuprate phase diagrams require a departure from the theory of Fermi liquids, a challenging property which can be realized in holography [6].

In the rest of this section we shall introduce the concepts and general properties of superconductivity which are needed to understand the holographic results discussed in the following sections, providing at the same time the reader with the organization of the paper and the conclusions. We assume the minimal field content to describe superconductivity: a charged scalar Φ_{cl} , responsible for the breaking of the U(1) symmetry and a gauge field a_μ . The quantum effective action is a generic functional of gauge invariant operators

$$\Gamma = \int d^3x \mathcal{L}_{\text{eff}}(\mathcal{F}_{\mu\nu}, \partial_\mu \arg(\Phi_{\text{cl}}) - a_\mu, \psi_{\text{cl}}) + \Gamma_{n.l.}, \quad (1)$$

where we have introduced the field strength $\mathcal{F}_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ and $\psi_{\text{cl}} = |\Phi_{\text{cl}}|$. The first term in the expression above represents the contribution of *local* gauge invariant terms. $\Gamma_{n.l.}$ depends instead on *non-local* gauge invariant objects, which may be required if the material has a non-trivial topology. Although the action in (1) is very general, its form simplifies in some limits. This allows us to extract model independent properties and, at the same time, identify the predictions of specific models, such as the holographic ones that will be discussed here.

One of these limits is the small field limit, in which Γ is approximated by the Ginzburg-Landau (GL) action:

$$\Gamma_{\text{GL}} = \int d^3x \left(-\frac{1}{4g_0^2} \mathcal{F}_{\mu\nu}^2 - |D_\mu \Phi_{\text{GL}}|^2 + \frac{1}{2\xi_{\text{GL}}^2} |\Phi_{\text{GL}}|^2 - b_{\text{GL}} |\Phi_{\text{GL}}|^4 \right), \quad (2)$$

where g_0 , ξ_{GL} and b_{GL} are real and positive parameters and we have rescaled the scalar field, $\Phi_{\text{GL}} \propto \Phi_{\text{cl}}$, to have a canonically normalized kinetic term for Φ_{GL} . Also $D_\mu \Phi_{\text{GL}} = (\partial_\mu - i a_\mu) \Phi_{\text{GL}}$. This theory holds if the system is close enough to the symmetry breaking transition, where Φ_{cl} is small. Another case in which the action simplifies is the limit of slowly varying fields, such that Γ can be approximated by a two-derivative functional

$$\Gamma \simeq \int d^3x h(\psi_{\text{cl}}) \left\{ -\frac{1}{4g^2(\psi_{\text{cl}})} \mathcal{F}_{\mu\nu}^2 - |D_\mu \Phi_{\text{cl}}|^2 - W(\psi_{\text{cl}}) \right\}, \quad (3)$$

where h , g and W are unspecified functions of ψ_{cl} . When (3) is a good approximation the SF limit can be taken as $g \rightarrow 0$: this corresponds to taking the gauge field non-dynamical and thus the corresponding U(1) symmetry global. Deep inside a uniform SC, where ψ is close to the minimum ψ_∞ of the potential $V = hW$ and the massive gauge field $a_\mu - \partial_\mu \arg(\Phi_{\text{cl}})$ is suppressed, (3) reduces to a functional that is quadratic in the fields; this allows us to define the inverse masses of ψ_{cl} and a_μ , respectively ξ and λ :

$$\frac{1}{\xi^2} = \frac{1}{2h(\psi_\infty)} \frac{\partial^2 V}{\partial \psi_{\text{cl}}^2}(\psi_\infty) > 0, \quad \lambda = \frac{1}{\sqrt{2e(\psi_\infty)} \psi_\infty}. \quad (4)$$

Note that ξ and λ cannot be computed in a model-independent way as their values strongly depend on the form of h , g and W . The distinctive property of the superconductive phase is $\psi_\infty \neq 0$, as opposed to the normal phase in which $\psi_\infty = 0$.

Starting from the general action in (1) it is possible to show in very general terms some of the most famous properties of SCs, such as the Meissner effect, the infinite DC conductivity and the Josephson effect [7], which have been obtained in holography in Refs. [4, 8, 9, 3, 10].

In this article (unless otherwise stated) we will focus on static configurations, in which everything is constant in time and the temporal component of the gauge field is identified with the chemical potential of the U(1) charge: $a_0 = \mu$. Also, in order to study the thermodynamics of these systems we introduce the free energy $F = -T\Gamma$.

In section 3 we will holographically study the normal phase and explain how to describe conductors and insulators. The homogeneous phase, in which $\psi_{\text{cl}} = \psi_\infty$, will be studied in section 4 through the AdS/CFT correspondence. As we will see, this provides a rationale for the fact that superconductivity is suppressed at large rather than at small T .

However, inhomogeneous configurations are also of great importance in SCs; e.g. they generically occur when the system is probed by an external EM field. Among the most famous properties of SCs we can certainly include the existence of vortex solutions, which are indeed strongly inhomogeneous configurations. The simplest example of such configurations are straight vortex lines, which can be described by an ansatz of the form $\Psi_{\text{cl}} = \psi_{\text{cl}}(r)e^{in\phi}$ and $a_\phi = a_\phi(r)$ and the other components of the vector potential set to zero. Here r, ϕ are the usual polar coordinates which parametrize the Euclidean two dimensional space. By using the action in (3) one obtains the following large r field behavior [4]

$$\psi_{\text{cl}} \simeq \psi_\infty + \frac{\psi_1}{\sqrt{r}} e^{-r/\xi'} , \quad a_\phi \simeq n + a_1 \sqrt{r} e^{-r/\lambda'} , \quad (5)$$

for SCs, while for SFs, where the magnetic field $B = \partial_r a_\phi / r$ is frozen to an external constant value, $a_\phi = Br^2/2$, the condensate approaches the homogeneous value with a power law $\psi_{\text{cl}} - \psi_\infty \sim n^2/r^2$ [4]. In Eq. (5) ψ_1 and a_1 are unspecified constants, while the action in (3) implies that $\lambda' = \lambda$ and $\xi' = \xi$. However, the exponential behavior in Eq. (5) tells us that higher derivative terms cannot be neglected as they are generically of the same order of the two-derivative terms. Fortunately, the only effect of the higher derivatives is to modify the values of λ' and ξ' , such that we generically have $\lambda' \neq \lambda$ and $\xi' \neq \xi$, as discussed in [4]. λ' and ξ' are respectively called penetration depth and coherence length and are important to characterize SCs. Although the field behavior in (5) is model independent, λ' and ξ' , like λ and ξ , can only be fixed once the model is specified. Thus their actual values are predictions of the particular model one considers; in section 5 we will discuss how to extract them from holography.

While vortex solutions exist for any SC, it is not always the case that there is a range of the external magnetic field H such that the vortex phase is energetically favorable; when this is true the SC is of Type II and such range is denoted with $H_{c1} \leq H \leq H_{c2}$ (if such range does not exist we have instead a Type I SC). H_{c2} is the value of the external magnetic field above which the system is always in the normal phase. The vortex configuration for Type II SCs and H slightly smaller than H_{c2} is known to be a triangular lattice of vortices independently on the specific model one considers: for those high magnetic fields the condensate is small and the GL theory can be applied to predict that configuration [11]. H_{c1} is instead the value of H below which the system is in the homogeneous superconducting phase. It can be computed through the model independent formula¹ (see e.g. [4])

$$H_{c1} = \frac{g_0^2}{2\pi} (F_1 - F_0) , \quad (6)$$

where F_1 and F_0 are the free energies for the $n = 1$ vortex and $n = 0$ superconducting phase respectively. We note that the actual value of H_{c1} is a prediction of the specific model one

¹ Here we use the normalization of H such that the external current \vec{J}_{ext} that generates H through $\nabla \times \vec{H} = g_0^2 \vec{J}_{\text{ext}}$ is coupled to the gauge field by means of the interaction term $a_\mu J_{\text{ext}}^\mu$ in the Lagrangian.

considers because g_0 , F_1 and F_0 are model dependent. However, in the SF limit, $g_0 \rightarrow 0$, we always have $H_{c1} \rightarrow 0$; in other words H_{c1} is non-trivial only if the magnetic field is dynamical.

In section 5 we will study the vortex phase in holography and illustrate how to compute the critical magnetic fields and show that the holographic SC is of type II. We will emphasize in particular the differences between the case in which there are no sources of conformal symmetry breaking, other than T and μ , and the case in which conformal symmetry is broken.

Up to now we have not discussed the role of the non-local terms in Eq. (1). For reasons which will be clear soon, we would like to spend the rest of this introductory section to describe the simplest setup in which $\Gamma_{n.l.}$ is relevant: cylindrical SCs threaded by an external magnetic field along the symmetry axis of the cylinder. To render the discussion even simpler we will take the deep Type II limit, $g_0 \rightarrow 0$, in which the (total) magnetic field coincides with the external one. Since there is a non-contractible loop on this geometry we can construct non-local gauge invariant objects:

$$W \equiv \exp \left(ei \oint dx^\mu a_\mu \right), \quad m \equiv \frac{1}{2\pi} \oint dx^\mu \partial_\mu \arg(\Phi_{cl}) = \text{integer}, \quad \dots, \quad (7)$$

where e is the charge of the fundamental charge carriers (say the electrons for real world SCs) and the integrals are performed along the non-contractible loop. We will refer to W and m as the Wilson line and the fluxoid respectively and the dots represent other non-local objects. While the classical action does not depend on (W, m, \dots) , quantum effects, such as the Aharonov-Bohm one (or Sagnac one [12, 13] in the SF literature), could introduce a dependence of Γ on these quantities. A simple (but still general) way to visualize a dependence of this sort is to think that the coefficients of the local part of Γ vary with (W, m, \dots) : for example, in the domain of validity of the GL theory in (2), this corresponds to thinking of ξ_{GL} and b_{GL} as functions of (W, m, \dots) . Such dependence ought to be suppressed in the classical limit and/or when the typical scale of the non-contractible loops, R , is large compared to the other scales in the problem (such as $1/T$ and $1/\mu$) and therefore we should stay away from these limits to see any interest effect. Moreover, *since quantum corrections are small in a weakly coupled theory, the biggest effects of the non-local quantities are expected in strongly coupled theories.* This justifies the use of holography in this setup (see section 6). Before moving to holography, however, let us identify model-independent effects.

We shall consider the simplest case in which the magnetic field is constant so that it can be represented by a constant vector potential along χ , which parametrizes the compact spatial dimension, $\chi \sim \chi + 2\pi R$. Since everything is static and homogeneous an appropriate ansatz is

$$\Phi_{cl} = \psi_{cl} e^{im\chi/R}, \quad a = \mu dt + a_\chi d\chi \quad \text{with } \psi_{cl} \text{ and } a_\chi \text{ constant.} \quad (8)$$

Inserting this ansatz in the quantum effective action, Eq. (1), we obtain

$$\Gamma = \int d^3x \mathcal{L}_{\text{eff}}(m/R - a_\chi, -\mu, \psi_{cl}) + \Gamma_{n.l.}; \quad (9)$$

in other words, *modulo non-local terms*, the system must be periodic in the magnetic flux² $\Phi(B) \equiv \oint dx^\mu a_\mu / g_0$ with period $\Delta\Phi^{LP} \equiv 2\pi/g_0$, because $\Phi(B) \rightarrow \Phi(B) + \Delta\Phi^{LP}$ can be compensated by a unit shift of the integer m . This is known as the Little-Parks effect [14]. This phenomenon has been observed in experiments, which give $g_0 = 2e$, and thus it is considered as an evidence for Cooper pairing. If the non-local terms in (9) are non-negligible the Little-Parks does not generically occur; indeed nothing forces $\Gamma_{n.l.}$ to be a function of the local combination $m/R - a_\chi$ only, but it may depend on m and W separately. The conclusion is that if the system is

² The factor $1/g_0$ has been introduced in order to have a canonically normalized kinetic term for a_μ in Eq. (2).

strongly coupled, far away from the classical limit and R is small enough we could see an uplifting of the Little-Parks periodicity to an enhanced periodicity, set by the fundamental charge: $2\pi/e$. This would correspond to resolving the internal structure of the *composite* condensing operator. Whether this really occurs is not model-independent.

Remarkably, we will see (in section 6) that holography *predicts* an uplifting of the Little-Parks periodicity for small values of R and that this uplifting corresponds in the gravity side to the celebrated Hawking-Page phase transition [15].

2. The holographic model

The first step to define a holographic model based on the standard AdS/CFT correspondence is to introduce fields living on an asymptotically AdS space. The minimal field content required to describe SCs holographically is a charged scalar field Ψ dual to a condensing operator \mathcal{O} ($\langle \mathcal{O} \rangle = \Phi_{\text{cl}}$) and a gauge field A_α dual to the electric current operator J_μ . As we have stated in the introduction, another ingredient we are interested in is a mechanism for conformal symmetry breaking in the IR; this can be achieved by introducing a real scalar ϕ (a *dilaton*) which acquires a non-trivial VEV. The class of actions we consider is [9]

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{16\pi G_N} [\mathcal{R} - (\partial_\alpha \phi)^2 - V(\phi)] - \frac{Z_A(\phi)}{4g_4^2} \mathcal{F}_{\alpha\beta}^2 - \frac{Z_\psi(\phi)}{L^2 g_4^2} |D_\alpha \Psi|^2 \right\}, \quad (10)$$

where \mathcal{R} is the Ricci scalar, G_N the Newton constant, V the dilaton potential, g_4 the U(1) gauge coupling and L is the radius of AdS. Also, Z_A and Z_ψ are generic functions of the dilaton, which do not fulfill particular properties, besides the fact that they never vanish (in order for the semiclassical approximation in the bulk to be justified). Also, $D_\alpha \Psi = (\partial_\alpha - iA_\alpha)\Psi$.

We are interested in describing a static system with two dimensional rotation and translation invariance, for which the metric and the dilaton have the form

$$ds^2 = W(z) \left(-f(z)dt^2 + dy^2 + \frac{dz^2}{f(z)} \right), \quad \phi = \phi(z), \quad (11)$$

where dy^2 is the two dimensional Euclidean metric. The temperature can be introduced by requiring the presence of a black hole; so there is a value of the holographic coordinate, z_0 , such that $f(z_0) = 0$. Then $T = |f'(z_0)|/4\pi$, where the prime denotes the derivative with respect to z .

Another physical situation that can be described with this setup is the case of multiply connected SCs, more precisely of cylindrical geometry: this can be simply achieved by compactifying one of the two y -coordinates: $\chi \sim \chi + 2\pi R$. Then there is necessarily another set of solutions [16]: those obtained from the black holes by exchanging the Euclidean time $t_E = it$ with χ ; these configurations have no event horizons and are commonly called AdS solitons. One reason the compactification of χ is interesting is because this procedure allows us to introduce a scale R (other than the T and μ) and thus breaks conformal invariance even in the absence of the dilaton. For this reason from now on the case in which χ is compact will be discussed with a simplified field content, where ϕ is removed and $Z_A = Z_\psi = 1$. The AdS soliton (black hole) geometry is energetically favorable with respect to the black hole (AdS soliton) at sufficiently small (large) temperatures, $T \leq 1/(2\pi R)$ ($T \geq 1/(2\pi R)$). Another reason why we are interested in compactifying a spatial coordinate is to study what happens when cylindrical SCs are threaded by magnetic fields: in these physical setups indeed a universal prediction of holography emerges, as it will be discussed in section 6 and anticipated in the introduction.

Although we include the possibility of conformal symmetry breaking in the IR, in this article we will always assume that the ultraviolet (UV) is conformally invariant, or in other words that the fields are approximated by an AdS configuration close to a value of z , say $z = 0$: for z close to

zero, $W(z) \simeq L^2/z^2$, $f(z) \simeq 1$ and $\phi(z) \simeq 0$. Thus we can use the standard AdS/CFT dictionary, which relates the properties of a gravitational theory with those of a CFT. In particular the $z = 0$ values of Ψ and A_μ are the sources of \mathcal{O} and J_μ in the CFT. Therefore, if one solves the bulk field equations with boundary conditions

$$\Psi|_{z=0} = \Psi_0, \quad A_\mu|_{z=0} = a_\mu, \quad (12)$$

the Green's function for \mathcal{O} and J_μ are given by differentiating the on-shell action with respect to Ψ_0 and a_μ ; e.g. the vacuum expectation values are given by

$$\langle J_\mu \rangle = \frac{1}{g_4^2} \mathcal{F}_{z\mu}|_{z=0}, \quad \langle \mathcal{O} \rangle = \frac{1}{g_4^2 z^2} D_z \Psi|_{z=0}. \quad (13)$$

In order to keep the discussion as simple as possible we assume that Ψ and A_α do not backreact on the geometry and the dilaton; this can be consistently achieved by taking the limit $G_N \rightarrow 0$. A generalization to the case $G_N \neq 0$ can be found in [17, 18]. A general form of asymptotically AdS dilatonic black hole solutions with the assumed symmetries was derived in [19]. One important aspect of these black holes is that the dilaton is not generically constant but runs with the energy:

$$\phi(z) = \sqrt{\frac{\nu^2 - 1}{2}} \ln(1 + z/L), \quad (14)$$

where ν is a real parameter of the potential such that $\nu \geq 1$ and $\nu = 1$ recovers the Schwarzschild black hole (S-BH) of Einstein's gravity. Eq. (14) tells us that although the theory has a UV fixed point, $\phi(0) = 0$, conformal invariance is broken and maximally in the IR, $z = z_0$. In order for the configurations of Ref. [19] to be a solution, V has to be appropriately chosen. In particular the requirement of an asymptotically AdS configuration space with cosmological constant Λ implies that at the conformal point (conventionally $\phi = 0$) where $V(0) = \Lambda$ we have $\partial V / \partial \phi = 0$. As a result, for each value of ν , even $\nu \neq 1$, the S-BH with cosmological constant Λ is always a solution. One finds [9] that the S-BH is favorable at high temperatures, while the dilaton-BH, Eq. (14), dominates in the low temperature region.

3. Conductivity

Having described the solutions of the dilatonic gravity systems we now want to understand the type of materials they correspond to. Some information can be gained by studying the conductivity, σ . This will also shed light on the nature of the SF phase transitions, which will be discussed in the next section.

To compute σ let us consider, on top of these geometries, a small plane wave along a spatial coordinate x ,

$$A_x(t, z) = \mathcal{A}(z) e^{i\omega(p(z)-t)}, \quad (15)$$

which is induced by a small electromagnetic field, $A_x(t, z)|_{z=0} = a_x(t)$. Here \mathcal{A} and p are real functions of z . The system responds creating a current which is linear in the electric field E_x : that is $\langle J_x \rangle = \sigma E_x$. Using the AdS/CFT dictionary, the first equation in (13), we have

$$\sigma = \frac{p'(0)}{g_4^2} - i \frac{\mathcal{A}'(0)}{g_4^2 \omega \mathcal{A}(0)}. \quad (16)$$

When an event horizon is present regularity of the solution implies that the plane wave should be ingoing rather than outgoing from the horizon. It can be shown that this results in a non-vanishing $\text{Re}[\sigma]$ and DC conductivity [9]

$$\lim_{\omega \rightarrow 0} \text{Re}[\sigma] = \frac{1}{g_4^2} Z_A|_{z=z_0} \neq 0. \quad (17)$$

For an AdS soliton, which has no event horizon, regularity allows for a vanishing DC conductivity, which has been identified with an insulating behavior [20, 18]. Therefore, compactifying a spatial dimension χ allows us to realize a conductor/insulator transition as the temperature is lowered. Eq. (17), however, tells us that there is another way to suppress the DC conductivity and obtain such transition if a dilaton is present in the spectrum: if one chooses Z_A sufficiently small for large values of ϕ , the running in Eq. (14) implies that $\lim_{\omega \rightarrow 0} \text{Re}[\sigma]$ is small, especially in the low temperature limit where z_0 is large. In this setup the transition (discussed in the previous section) between the S-BH and dilaton-BH, which is obtained by lowering the temperature, represents a conductor/insulator transition. It is worth noting that insulating systems corresponds to solids in the fluid mechanical interpretation [8].

4. Superfluid phase transition

We now move on and study the simplest example of superconducting state: a static, homogeneous and isotropic material in which the $U(1)$ symmetry is spontaneously broken. The ansatz is

$$\Psi = \psi(z), \quad A = A_0(z)dt. \quad (18)$$

The reader may wonder why we have introduced the temporal component of the gauge field, because having a non-vanishing $\psi(z)$ seems already enough to describe the state we are interested in. The reason is that no regular solution is found when $A_0 = 0$ in (18), as shown in [9], and we want to exclude singular solutions. Physically this corresponds to the fact that there is only one parameter breaking conformal symmetry (T for the black hole and R for the AdS soliton) in the absence of the dilaton³, thus no phase transition can occur in this case. Therefore we set the UV boundary conditions

$$\Psi_0 = 0, \quad a_0 = \mu. \quad (19)$$

The first condition guarantees that the $U(1)$ symmetry is spontaneously broken, while the second one, with a non-vanishing chemical potential μ , keeps the profile of A_0 different from zero and allows us to find a regular solution.

An immediate consequence of this argument is that when the temperature is large compared to the chemical potential a black hole suppresses the superconductivity. At small temperatures instead the system turns into a SC [3]. In AdS/CFT there is therefore a reason for the fact that a metal becomes a SC at small rather than at large temperatures. For the AdS soliton the same conclusion can be reached, but with T substituted by $1/R$.

According to the results of the previous section these SF phase transitions can be either of the conductor/SC or insulator/SC type, depending on the behavior of the conductivity in the normal phase: the two cases correspond to a non-vanishing or vanishing DC conductivity respectively. In the fluid mechanical interpretation, the SF phase of a holographic system with a solid normal phase has been identified with a supersolid [21].

5. Dynamical gauge fields in AdS/CFT and superconductivity

In the holographic results we have discussed so far the dynamics of the EM field is not important. Therefore, according to the discussion presented in the introduction, they can be applied equally well to SFs and SCs. In this section instead we discuss some important effects of superconductivity, which crucially rely on the dynamics of the EM field.

However, we note that imposing the first boundary condition in (12) treats a_μ as an external source, which does not participate in the dynamics of the system. In order to have a dynamical

³ The dilaton cannot change this conclusion as it is regular for any T and the function Z_ψ , Eq. (10), is assumed to be regular and non-zero.

a_μ we should integrate over all possible field configurations: (working in the Euclidean space)

$$Z[J_{ext}] = \int Da e^{-S_E[a_\mu] + \int d^3x \left(-\frac{1}{4g_b^2} \mathcal{F}_{\mu\nu}^2 + a_\mu J_{ext}^\mu \right)}, \quad (20)$$

where $S_E[a_\mu]$ is the Euclidean bulk action computed on a solution of the field equations with boundary condition in (12). Also, we have introduced for generality a kinetic term for a_μ and an external current J_{ext} . Then $Z[J_{ext}]$ defines as usual the generating functional for the Green functions of a_μ . Of course, if there are other operators in the theory, besides the gauge field, $S_E[a_\mu]$ in Eq. (20) will depend on the corresponding sources as well; for example in the case of superconductivity discussed in the previous sections $S_E[a_\mu]$ also depends on Ψ_0 , which we introduced in (12).

In the semiclassical limit this procedure reduces to solving the Maxwell equations,

$$\langle J^\mu \rangle + \frac{1}{g_b^2} \partial_\nu \mathcal{F}^{\nu\mu} + J_{ext}^\mu = 0, \quad (21)$$

where we have used $\langle \hat{J}^\mu \rangle = -\delta S_E / \delta a_\mu$. The semiclassical limit corresponds to taking a large external current, $g_b \rightarrow 0$ and a limit on the parameters of the bulk theory such that $S_E[a_\mu]$ becomes large; for example, for the theory defined in (10) this limit is $G_N \rightarrow 0$ and $g_4 \rightarrow 0$.

Now, using Eq. (13), we can see that the Maxwell equations in (21) can be viewed as a Neumann type boundary condition in the gravity side:

$$\frac{1}{g_4^2} \mathcal{F}_z{}^\mu \Big|_{z=0} + \frac{1}{g_b^2} \partial_\nu \mathcal{F}^{\nu\mu} \Big|_{z=0} + J_{ext}^\mu = 0. \quad (22)$$

The lesson is therefore that switching from the Dirichlet boundary condition (12) to the condition above promotes a_μ to a dynamical field.

As far as the holographic SC model of section 2 is concerned, this procedure has been applied to find the Meissner effect and genuine SC vortices [4, 9], at least in the simplest case of a straight vortex line:

$$\Psi = \psi(z, r) e^{in\phi}, \quad A_0 = A_0(z, r), \quad A_\phi = A_\phi(z, r). \quad (23)$$

r, ϕ are the usual polar coordinate parametrizing the two dimensional Euclidean space and so here we assume that there are at least two non-compact dimensions in the CFT. This is not the case for the four dimensional AdS soliton; however, vortex solutions have been found on top of the five dimensional AdS soliton in [8]. Using the ansatz in (23) the Neumann-like boundary condition in (22) becomes

$$\frac{1}{g_4^2} \partial_z A_\phi \Big|_{z=0} + \frac{1}{g_b^2} r \partial_r \left(\frac{1}{r} \partial_r A_\phi \right) \Big|_{z=0} + J_{ext} \phi = 0, \quad (24)$$

while the requirement of spontaneous symmetry breaking and the presence of a finite charge density again fixes the other $z = 0$ boundary conditions, Eq. (19). Regularity of the solutions instead fixes the conditions at $z = z_0$ and at the center of the vortex $r = 0$ and physical conditions on the behavior at infinity, $r \rightarrow \infty$, imposes constraints on the remaining boundary.

This results in genuine SC profiles for the total magnetic field $B = \partial_r a_\phi / r$: for example for $n = 0$ one recovers the Meissner effect, while for $n \neq 0$ one observes the exponential damping of B far away from the center of the vortex [4, 8, 9], Eq. (5), allowing for a holographic prediction for λ' and ξ' . Such properties should be contrasted with the Dirichlet boundary condition for the

magnetic field, which in polar coordinates is $A_\phi|_{z=0} = Br^2/2$ (with B constant), corresponding to SF systems (see e.g. [21, 4, 8, 9, 22]).

Once the vortex solutions are obtained we can compute H_{c1} through Eq. (6). Indeed selecting the $n = 1$ vortex and $n = 0$ homogeneous superconducting phase we can compute $F_1 - F_0$ while, integrating over the bulk *à la* Kaluza-Klein gives g_0 :

$$\frac{1}{g_0^2} = \frac{1}{g_4^2} \int_0^{z_0} dz Z_A(\phi). \quad (25)$$

The fact that the dilaton appears in this formula has an important consequence on the behavior of H_{c1} at low temperatures [9], as we now explain. In the absence of ϕ Eq. (25) implies that $g_0 \rightarrow 0$ as $T \rightarrow 0$ (in this limit the horizon is removed, $z_0 \rightarrow \infty$) and Eq. (6) forces H_{c1} to vanish. The physical reason why this emerges is because the theory without the dilaton is scale invariant and so g_0 should go to zero as $T \rightarrow 0$. The presence of the dilaton breaks scale invariance and can avoid this conclusion, providing a non-vanishing H_{c1} : g_0 remains non-zero all the way down to $T = 0$ if $Z_A(\phi)$ goes to zero fast enough as $\phi \rightarrow \infty$, a limit that always occurs close to the horizon as $T \rightarrow 0$, Eq. (14). Interestingly, when H_{c1} remains non-zero at zero temperature, Eq. (16) tells us that the corresponding normal phase must have a suppressed DC conductivity in the low temperature region, resembling an insulator.

The second critical field H_{c2} can be instead computed as the value of H at which the condensate goes to zero. Up to now all the holographic models of superconductivity have turned out to be of Type II [4, 8, 9], like all known high temperature SCs.

6. Multiply connected holographic superconductors and superfluids

As explained in section 1 an interesting setup to apply the holographic techniques are multiply connected SCs probed by magnetic fields. We shall consider the simplest case described at the end of section 1. In the presence of a compactified dimension scale invariance is broken and we will assume for simplicity that the bulk field content does not include the dilaton. Also, the ansatz in (8) corresponds holographically to

$$\Psi = \psi(z)e^{im\chi/R}, \quad A_0 = A_0(z), \quad A_\chi = A_\chi(z), \quad (26)$$

with UV boundary conditions $\Psi|_{z=0} = 0$, $A|_{z=0} = \mu$, $A_\chi|_{z=0} = a_\chi$. As already discussed, the system is in the S-BH phase at large radii and in the AdS soliton otherwise and, as shown in [23], such transition corresponds to an uplifting of the Little-Parks periodicity. This is a universal prediction of holography as it uniquely relies on the geometrical properties of the S-BH and the AdS soliton. The simplest way to understand this point is to look at the IR boundary condition (at $z = z_0$) for A_χ and ψ , which regularity of the bulk solutions requires:

$$A'_\chi + \frac{2\psi^2}{3z_h} \left(A_\chi - \frac{m}{R} \right) \Big|_{z=z_h} = \frac{3\psi'}{z_0} + \left(A_\chi - \frac{m}{R} \right)^2 \psi \Big|_{z=z_0} = 0 \quad (27)$$

for the S-BH and

$$A_\chi|_{z=z_0} = 0, \quad \left\{ \begin{array}{ll} \frac{3\psi'}{z_0} - A_0^2 \psi \Big|_{z=z_0} = 0 & \text{for } m = 0, \\ \psi|_{z=z_0} = 0 & \text{for } m \neq 0 \end{array} \right. \quad (28)$$

for the AdS soliton. While the conditions in (27) only depend on the local combination of m and W , that is $m/R - a_\chi$, the conditions in (28) depend separately on the fluxoid and the Wilson line. This means that the non-local terms in the quantum effective action are suppressed

(unsuppressed) in the S-BH (AdS soliton) phase. Correspondingly, according to the model-independent discussion of section 1, the system should be characterized by the Little-Parks periodicity for the S BH and an uplifting of such periodicity for the AdS soliton as shown in [23, 8].

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